Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Networked Systems Group (NSG)

HS 2026

Prof. L. Vanbever / J. Mégret
based on Prof. R. Wattenhofer's material

Discrete Event Systems

Exercise Sheet 6

1 Quiz Questions

a) Can a sequence of random variables, where each random variable depends on the 2 previous random variables, be described by a Markov chain?

b) Is the following statement true? If a finite Markov chain is irreducible and there is a state which has period 1, then the Markov chain is ergodic.

c) If, in a Markov chain, there is a state with period 2 and a state with period 3, then the Markov chain is always which of the following? Irreducible, not irreducible, ergodic, aperiodic.

d) For each of the links between u, v and w, give an intuitive argument as to whether removing the link will improve or deteriorate the PageRank of u?

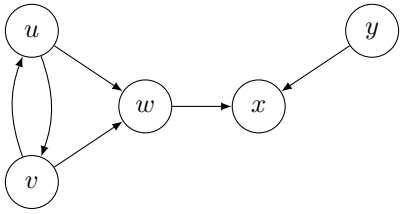


Figure 1: An example of a web graph with 5 websites. Website x does not link to any other website, i.e., x is a *sink*.

e) Is the following statement true? By performing a simple random walk starting at any vertex for a long enough time, the probability of being in a specific vertex gets arbitrarily close to the unique stationary distribution probability of that vertex.

2 Soccer Betting

The *Nati* (swiss national football team) is a particularly moody team. A group of international scientists, consisting of soccer experts, mathematicians, and psychologists, has recently conducted a thorough analysis of this behavior. They have discovered that upon winning a game, the Nati wins the next game with a probability of 0.6 as well. With probabilities 0.2 each, the next game will be a tie or a loss. After a loss, the Nati will win/tie/lose its next game with probability 0.1/0.2/0.7, respectively. Finally, after a tie, the next game being a win or a loss is equally probable. The probability that the next game also ends up being a tie is 0.4.

a) Model the Nati's moodiness using a Markov chain.

b) In *two games* from now (they will play one game in between), the Nati will play against *les Bleus*. Bookmakers offer you the following odds:

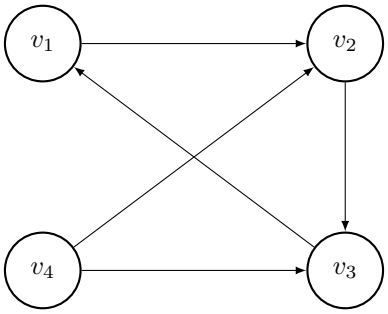
Win: 3.5 Tie: 4.0 Loss: 1.5

Given that the Nati won the game three games ago, but lost the last two games, what would be your bet? Why?

c) More recent studies have shown that the Nati is even moodier than expected. In fact, after losing *two games in a row*, the probability of winning its next game reduces to 0.05, that of getting a tie to 0.1. Change your Markov chain model to incorporate the new findings. How does the change influence your bet?

3 PageRank

Consider the following mini-internet with four websites and hyperlinks between those websites represented by directed links. In this exercise, we explore ways to compute a ranking of those websites, a PageRank, and consider how fair they are.



a) A naïve suggestion is to use the nodes' in-degree as a rank. This can be done in general by computing $(1,1,1,1) \cdot A$, where A is the adjacency matrix of the graph. What are the naïve PageRanks? Do you see any issues with this ranking?

b) An improved-naïve version would normalize each non-zero row in the adjacency matrix by dividing it by the sum of its entries. Denote the obtained matrix by A'. What are the improved-naïve PageRanks $(1,1,1,1) \cdot A'$? Is the problem from **a)** solved now?

c) Since websites with higher rank should have a larger impact on the rank of the nodes they link to, you decide to iterate the process from **b)**. Denoting $(1,1,1,1)$ with q_0, calculate the iterated version of this "PageRank" by computing

$$q_1 = q_0 \cdot A'$$
$$q_2 = q_1 \cdot A'$$
$$q_3 = q_2 \cdot A'$$
$$\vdots$$

(You should only need to iterate very few times.) When does the rank converge, and what is the issue? Would the same happen when using the Google matrix?

4 Chess Walk

Assume we have an empty 8×8 squares chess board.

a) If a Knight (Springer) is starting from one of the 4 middle squares of the board and moves by picking the next square uniformly at random from its set of legal moves, how many moves on average will it take for it to return back to the same square?[1]
$Hint_1$: The "degree sum theorem" says that the sum of degrees of the nodes of an undirected graph is equal to twice the number of edges in the graph.
$Hint_2$: If a knight's legal moves is represented as an undirected graph with edges between vertices (squares on the board) that can reach each other with a single move, the degree of a vertex $\delta(u)$ is simply the number of legal moves. For example, the 4 corner nodes have $\delta(u) = 2$, and so on.

b) If a Knight starts at one of the corners, does the answer change? If so, to what?

c) If a Rook (Turm) is placed on the bottom left corner a smaller 2×2 chess board, and moves by picking the next square uniformly at random from its set of legal moves, how many moves on average will it take for it to reach the top right corner of the board?

5 Queues

Router R, working for its favorite network, is very reliable: In each time step, it forwards exactly one packet from its queue. It likes the network so much because in each time step there arrive 0, 1 or 2 packets and each possibility occurs with the same probability. If, at time t, R's queue is empty, but at least one packet arrives in the time step between t and $t+1$, it forwards one of the arriving packets immediately. The queue can accommodate up to 10 packets. If, at time t, the queue is full, R forwards one of the contained packets as usual, but it also fills the queue to its limit again, if a new packet should arrive between t and $t+1$.

a) Model the load of the queue by using a Markov chain. What is the stationary distribution?

Hint: Try to rewrite the definition of stationary distribution (Definition 4.10 in the script) as a system of linear equations. Make use of the fact that most entries are 0.

b) Due to a magic upgrade, R's queue now can accommodate all packets that ever arrive. What is the stationary distribution now (for the initial probabilities for the number of arriving packets)? Can you see a problem with that?

c) R has become old. The magic has disappeared and its maximum queue size is back to 10 packets. Moreover, it is annoyed by colleagues sending packets when they know that its queue is full. Thus, if its queue is full at time t and 2 packets arrive between t and $t+1$ it simply flushes its queue. How does the stationary distribution change?

d) R has become very old. It stopped flushing out of frustration, but now considers retiring to some island, but since it still likes the work it does, it lets chance decide about the time of its retirement. In order to not retire too soon, it chooses to throw a very biased coin each time its queue is full and 2 packets arrive: If the coin shows head (which happens with probability 0.000001) it retires immediately, otherwise it continues its work. What does the stationary distribution of the queue look like now?

[1] A knight's move is L-shaped. It can jump by first moving in a direction by 2 squares either horizontally or vertically, and then move one square perpendicularly to the 2-square movement.