Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
Networked Systems Group (NSG)

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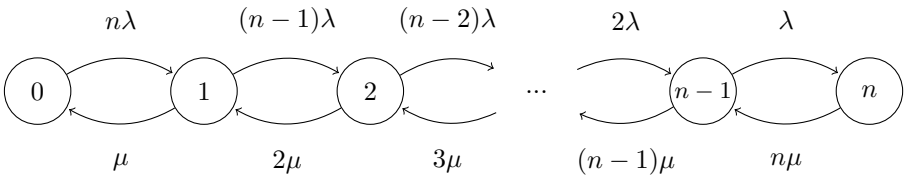
Prof. L. Vanbever / J. Mégret
based on Prof. R. Wattenhofer's material

Discrete Event Systems

Exercise Sheet 7

1 Üebigstund

a) We know that the minimum of i independent and exponentially distributed (with parameter λ) random variables is an exponentially distributed random variable with parameter $i\lambda$. Thus, we have the following birth-death-process:



b) First of all, note that we are working with a *finite irreducible CTMC*. Then, Theorem 4.12 from the lecture notes guarantees that we have a stationary distribution.

Let π_i be the probability of state i in the equilibrium. For $i \geq 1$, we know that

$$\pi_i = \pi_0 \cdot \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}$$

and thus

$$\pi_i = \pi_0 \cdot \frac{\lambda_0 \cdot \lambda_1 \cdot \cdots \cdot \lambda_{i-1}}{\mu_1 \cdot \mu_2 \cdot \cdots \cdot \mu_i}.$$

Applying this formula to our process yields

$$\pi_i = \pi_0 \cdot \frac{n(n-1) \cdot \cdots \cdot (n-i+1) \cdot \lambda^i}{1 \cdot 2 \cdot \cdots \cdot i \cdot \mu^i} = \pi_0 \cdot \binom{n}{i} \cdot \rho^i$$

where $\rho := \frac{\lambda}{\mu}$.

Note that we can write a similar term for π_0.

$$\pi_0 = \pi_0 \cdot 1 = \pi_0 \cdot \binom{n}{0} \cdot \rho^0$$

As the sum of all probabilities equals 1, we obtain the following.

$$\sum_{i=0}^{n} \pi_i = \pi_0 \sum_{i=0}^{n} \binom{n}{i} \cdot \rho^i = 1$$

Using the given formula for the binomial series

$$\sum_{i=0}^{n} \binom{n}{i} \cdot x^i = (1+x)^n,$$

we obtain

$$\pi_0(1+\rho)^n = 1 \implies \pi_0 = \frac{1}{(1+\rho)^n} \ .$$

Using the equations above for $i \geq 1$, we obtain

$$\pi_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n} \ .$$

c) (i) We have $\rho = 3/9 = 1/3$. Then, we calculate the probability that there are less than two students showing up.

$$\begin{aligned}
\pi_0 + \pi_1 &= \frac{1}{(1+\rho)^n} \cdot \left(1 + \binom{n}{1} \cdot \rho^1\right) \\
&= \left(\frac{3}{4}\right)^5 \cdot \left(1 + \frac{5}{3}\right) \\
&= \frac{3^5}{2^{10}} \cdot \frac{8}{3} \\
&= \frac{3^4}{2^7} \approx 0.63
\end{aligned}$$

Thus, the Üebigstund is cancelled with probability 0.63.

(ii) Now, $\rho = 3/4$. Again, we calculate $\pi_0 + \pi_1$.

$$\begin{aligned}
\pi_0 + \pi_1 &= \frac{1}{(1+\rho)^n} \cdot \left(1 + \binom{n}{1} \cdot \rho^1\right) \\
&= \frac{4^5}{7^5} \cdot \left(1 + 5 \cdot \frac{3}{4}\right) \\
&= \frac{4^4 \cdot 19}{7^5} \approx 0.29
\end{aligned}$$

Hence, the probability that the Üebigstund is cancelled is only 0.34!

(iii) In general, if $\rho \geq 1$, an M/M/1 queue might grow infinitely and therefore doesn't have a stationary distribution. This cannot happen in this birth-and-death process, though, because there is only a bounded number of states. Hence, the process has a stationary distribution even for $\rho \geq 1$.

2 Theory of Ice Cream Vending

The situation can be described by a classic M/M/2 system. There is an equilibrium iff

$$\rho = \lambda/(2\mu) < 1 \ .$$

For the stationary distribution and $m = 2$, it holds that

$$\begin{aligned}
\pi_0 &= \frac{1}{\left(\sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!}\right) + \frac{(\rho m)^m}{m!(1-\rho)}} \\
&= \frac{1}{\frac{(2\rho)^0}{0!} + \frac{(2\rho)^1}{1!} + \frac{(2\rho)^2}{2!(1-\rho)}} \\
&= \frac{1}{1 + 2\rho + \frac{4\rho^2}{(2(1-\rho))}} \\
&= \frac{1}{1 + 2\rho + \frac{4\rho^2}{2(1-\rho)}} \\
&= \frac{1}{\frac{2(1-\rho)+4\rho(1-\rho)+4\rho^2}{2(1-\rho)}} \\
&= \frac{2(1-\rho)}{2 - 2\rho + 4\rho - 4\rho^2 + 4\rho^2} \\
&= \frac{2(1-\rho)}{2 + 2\rho} \\
&= \frac{1-\rho}{1+\rho} \ .
\end{aligned}$$

3 Gloriabar

a) You might be tempted to model this situation by a queue with a bounded number of states, because the maximal number of persons in the line is bounded by 540. However, the situation can also be modeled by an infinite M/M/1 queue without losing too much accuracy; the parameter ρ will not be too large, such that the probability to reach the state in which 540 persons are standing in the queue at once is extremely small anyway. The modeling by an infinite M/M/1 queue conveniently allows us to apply Little's Law and therefore, we can use the formulae for the response and waiting time: Our arrival rate λ and service rate μ are given by

$$\lambda = \frac{540}{90 \cdot 60} = \frac{1}{10}, \quad \mu = \frac{1}{9}$$

(persons per second). Thus $\rho = \frac{\lambda}{\mu} = \frac{9}{10}$. By Theorem 4.21, the expected number of persons in the M/M/1 system is

$$N = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda} = 9.$$

By applying Little's Law, we learn that the expected time until the student has paid for her food is $T = \frac{N}{\lambda} = 90$ seconds. The expected waiting time is $W = T - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda} = 81$ seconds.

b) We use the formula for the expected number of jobs in the queue and obtain queue length of $N_Q = \frac{\rho^2}{1-\rho} = 8.1$.

c) We again use Little's Law with $T = \frac{N}{\lambda} = \frac{1}{\mu - \lambda} = \frac{90}{2}$, where $\lambda = 0.1$. Thus, $\mu = \frac{11}{90}$, i.e., instead of 9 secs, the service time should be $\frac{90}{11} \approx 8.2$ secs.