Stochastic processes

Markov Chains and Queuing Theory



Andrey Markov (1856–1922)

Photo: History of the USSR Academy of Sciences, 1964 (public domain)

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Part 7

Last week we introduced

Markov Chains

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Markov Chains

A stochastic process without memory

the next state depends *only* on the current state

Definition: Markov Chain

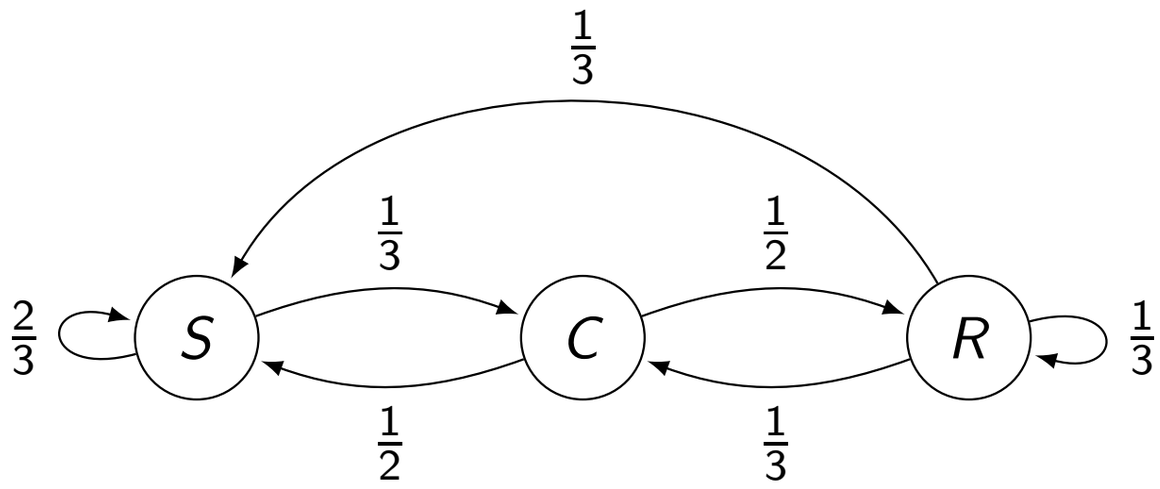
- Let S be a finite or countably infinite set of states.
- A discrete-time **Markov chain** is a sequence of random variables $X_0, X_1, X_2, \ldots \in S$ with the **Markov property**: X_{t+1} depends only on X_t, not on the past $X_{t-1}, \ldots, X_0$ ("memoryless"):

$$\Pr[X_{t+1} = s_{t+1} \mid X_0 = s_0, \ldots, X_t = s_t] = \Pr[X_{t+1} = s_{t+1} \mid X_t = s_t]$$

  for all t and all $s_0, \ldots, s_{t+1} \in S$.
- **Note:** memoryless is (usually) a simplification.
- The distribution of X_0 is the **initial distribution** $q_0 = (q_{0,s})_{s \in S}$ with $q_{0,s} = \Pr[X_0 = s]$.
- Time is discrete ($t \in \mathbb{N}_0$), for now. Continuous time ($t \in \mathbb{R}_0^+$) is the topic of the next chapter.

A First Example: The Weather in Zürich

- Each day is sunny (S), cloudy (C), or rainy (R).
- **Key assumption:** tomorrow's weather depends only on today's, with fixed probabilities:



An edge $a \to b$ is labelled with $\Pr[\text{tomorrow} = b \mid \text{today} = a]$.

The Transition Matrix and Initial Distribution

A handy way to collect the probabilities is through a **transition matrix** P:

$$P = \begin{array}{c|ccc} & S & C & R \\ \hline S & 2/3 & 1/3 & 0 \\ C & 1/2 & 0 & 1/2 \\ R & 1/3 & 1/3 & 1/3 \end{array}$$

- Entry $p_{i,j} = \Pr[\text{tomorrow} = j \mid \text{today} = i]$.
- Each row necessarily sums to 1.

- Alongside P, we typically also fix an **initial distribution** q_0 over the states, e.g. $q_0 = (1, 0, 0)$ if today is sunny.

We then used Markov Chains to analyze practical properties of stochastic systems

1 Sojourn time

2 Hitting time/Arrival probability

3 Stationary distribution

Sojourn Time

- It is sunny today, how many days will it *stay* sunny?
- The number of time steps the chain stays in state i (from time t on) is a random variable: the **sojourn time** T_i.
- It holds that $\Pr[T_i = k] = p_{i,i}^{k-1}(1 - p_{i,i})$ and $\Pr[T_i > k] = p_{i,i}^{k}$:
  T_i is **geometrically distributed**, so $\mathbf{E}[T_i] = \frac{1}{1-p_{i,i}}$.
- **Note:** The sojourn time is independent of how long the chain has already been in state i (memoryless).
- E.g. the weather chain stays sunny for $\frac{1}{1-2/3} = 3$ days on average; cloudy has no self-loop ($p_{C,C} = 0$), so it is left after a single day.

Computing the Hitting Time and Arrival Probability

Lemma

*For the **expected hitting time**, for all $i, j \in S$ it holds*

$$h_{i,j} = 1 + \sum_{k \neq j} p_{i,k} h_{k,j},$$

if $h_{i,j}$ and $h_{k,j}$ exist.

*For the **arrival probabilities**, analogously,*

$$f_{i,j} = p_{i,j} + \sum_{k \neq j} p_{i,k} f_{k,j}.$$

Ergodic Markov Chains

- **Finite**, irreducible, and aperiodic Markov chains are called **ergodic**.

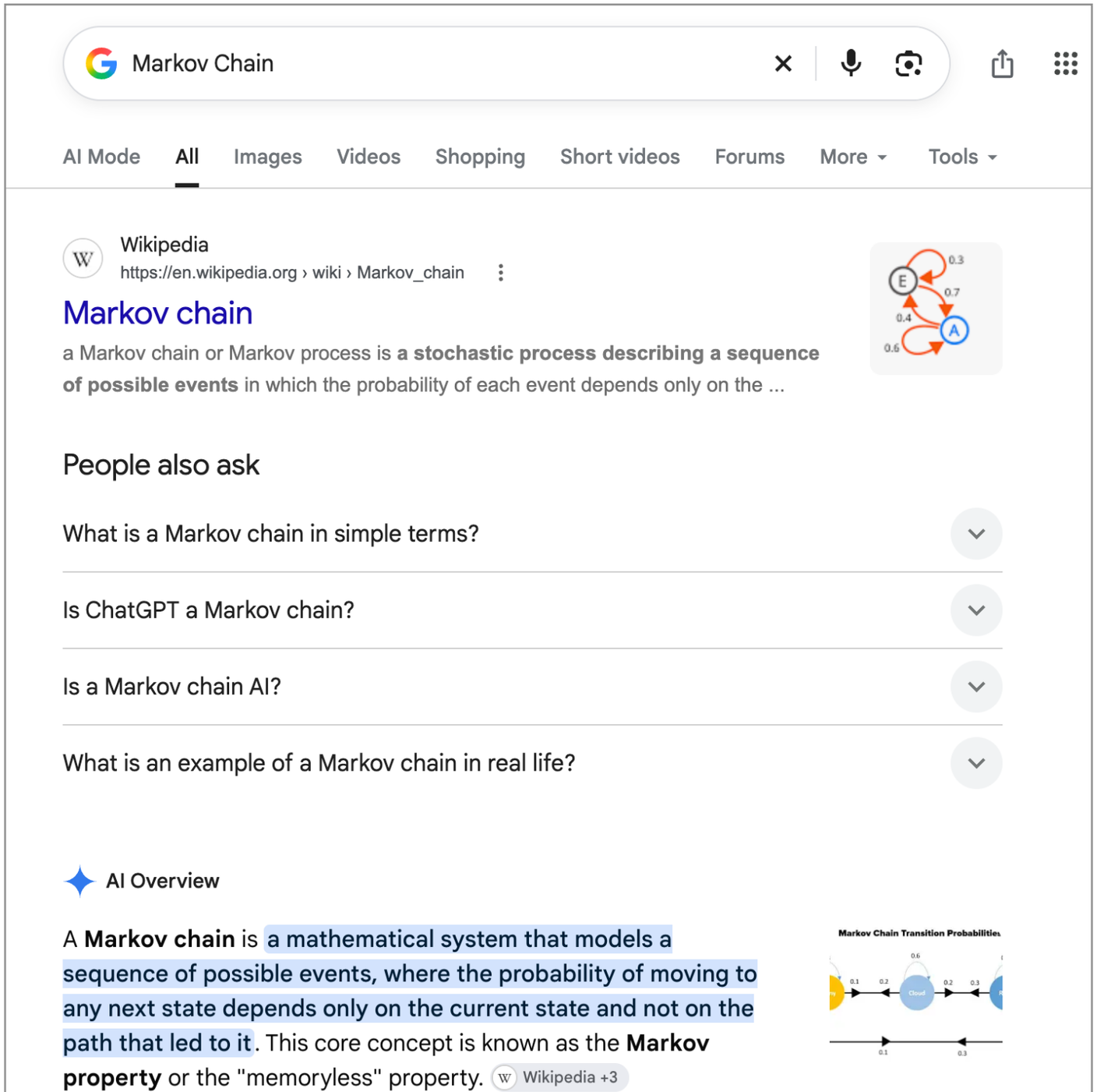
Theorem (Fundamental Theorem for Ergodic Markov Chains)

For every ergodic Markov chain, independently of the initial state,

$$\lim_{t\to\infty} q_t = \pi,$$

where π is the unique stationary distribution of the chain.

Finally we started to look at Google's PageRank algorithm which is used to rank search results



This week we introduce Continuous-Time Markov chains and start to look at Queuing Theory

Today's plan

Thu Oct 29

1 Continuous-time Markov chains

2 Queuing theory: The beginning